Maxwell's equations with Complex electric and magnetic fields due to massive photon
MAXWELL’S EQUATIONS WITH COMPLEX ELECTRIC AND MAGNETIC FIELDS DUE TO MASSIVE PHOTON

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Abstract—By defining complex electric and magnetic fields yields two set of Maxwell’s equations are obtained. The first set gives Maxwell’s equations in free space, and the second set gives Maxwell’s equations in a medium. These complex fields are assumed to be associated with massive photons. The complex Maxwell’s equations are shown to be invariant under duality transformation. Furthermore, two wavefunctions describing massive photons are introduced. One describes the electric charge and the second describes the magnetic charge due to massive photons. The two wavefunctions are attributed to the electromagnetic field due to electric and magnetic charges.

Keywords: Lorentz transformation; Maxwell’s equations; Magnetic charge; duality transformation; parity transformation

1. INTRODUCTION

Recently Dvoeglazov and Gonzalez expressed the Lorentz transformation of electric and magnetic field relating the $\gamma = E/mc^2$ factor (where $E$ is the energy) to the particle mass ($m$) [1]. Hence, they obtained the transformed electric and magnetic fields involving the mass of the particle due to Lorentz. By adopting the notion that photons are massive and charged, we have derived the electric and magnetic fields produced by these massive photons [2]. These fields are analogous to those obtained by Dvoeglazov and Gonzalez. However, the mass term in our formulation is related to the photon mass, and not to the particle mass as assumed by Dvoeglazov and Gonzalez [2]. Maxwell’s theory governs the electromagnetic field due to an electric charge. A static electric charge produces an electric field, and a moving electric charge gives rise to magnetic field. The evolution of the two fields is described by Maxwell’s equation. No notion of a magnetic charge is incorporated in Maxwell’s theory. The theory denies the presence of magnetic charge (monopole). In the framework of quantum mechanics, Dirac showed that a magnetic charge can exist, and ushers into the direction of quantization of the electric charge [3].
Furthermore, the electromagnetic interactions are carried by massless photon. However, the possibility of mediating the interaction with massive photons was investigated by Proca [4]. If the photon is massive, we can associate a magnetic charge that produces both electric and magnetic fields. Analogous to electric charge, an stationary magnetic charge has a magnetic field and produces an electric field when moves. The Maxwell’s equations describing the evolution of electromagnetic fields associated with magnetic charge can be obtained from the one associated with electric charge (ordinary Maxwell’s equations) by applying the duality transformations to the fields and currents. The electromagnetic wave produced from an oscillating electric charge or magnetic charge can’t be distinguished.

In this note we express the electric and magnetic fields in complex form. We are inspired by the Lorentz transformation of the electric and magnetic fields, as performed by Dvoeglazov and Gonzalez. This leads to the presence of a magnetic force acting on a magnetic charge reflecting the symmetry between the electric and magnetic phenomena. Choosing the mass term to be due to the mass of the photon, the magnetic charge of the photon can be seen to emerge from the complex electric and magnetic fields associated with massive photons.

It is customary to express Maxwell’s equations in terms of the complex electromagnetic vector, $\vec{F} = \vec{E} + ic\vec{B}$ [5, 6]. We however follow in this note a rather different approach.

2. MAXWELL’S EQUATIONS IN NEW FIELDS

The transformation of the electric and magnetic fields are recently studied by Dvoeglazov and Gonzalez [1] with \((c = 1)\). They follow a pattern of the form

$$\vec{E}' = \alpha \vec{E} - i\lambda \vec{B}, \quad \vec{B}' = \alpha \vec{B} + i\lambda \vec{E},$$  \hspace{1cm} (1)

where

$$\alpha = \gamma - \frac{\gamma^2}{\gamma^2 + 1} \left( v^2 - (\vec{S} \cdot \vec{v})^2 \right), \quad \lambda = \gamma(\vec{S} \cdot \vec{v}),$$ \hspace{1cm} (2)

and $\vec{S}$, $\vec{v}$ and $\gamma$ are the spin, velocity vectors and Lorentz factor, respectively. Let us consider eq.(1) as expressing some auxiliary fields. We assume here these fields are applicable for massive photons. The Lorentz force on a moving charge $q_e$ that is given by

$$\vec{F}_e = q_e(\vec{E} + \vec{v} \times \vec{B}),$$ \hspace{1cm} (3)

will be transformed into

$$\vec{F}_e' = \alpha q_e(\vec{E} + \vec{v} \times \vec{B}) - i\lambda q_e(\vec{B} - \vec{v} \times \vec{E}),$$ \hspace{1cm} (4)

upon substituting eq.(1) in eq.(3). Recall that in Maxwell’s theory the electric charge is the source of the electric field, while the magnetic field is not connected with magnetic charge. However, a moving electric charge produces magnetic field. If we now dictate
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a comprehensive duality symmetry between the electric and magnetic fields, a magnetic charge, that is a source of the magnetic field, should exits and creates electric field when it moves. Equation (4) can be expressed as

$$\vec{F}_e' = \alpha \vec{F}_e - i\lambda \vec{F}_m,$$

where

$$\vec{F}_m = q_e(\vec{B} - \vec{v} \times \vec{E}).$$  

Now apply eq.(1) in eq.(6) to obtain

$$\vec{F}_m' = \alpha \vec{F}_m + i\lambda \vec{F}_e.$$

Notice that $\vec{F}_e$ follows $\vec{E}'$ and $\vec{F}_m$ follows $\vec{B}'$ format. This formulation agrees with our recent formulation obtained by considering complex Maxwell’s equations [7]. It suggests that the electric and magnetic charges are not independent but related by the duality transformations. The massive photon can’t exhibit electric and magnetic charges simultaneously as these two properties are connected by the uncertainty relation $q_e q_m = n \frac{\hbar}{2}$ [3]. The electric and magnetic charges are thus coupled to each other and no longer independent. The magnetic charge produces electromagnetic effects analogous to those produced by the electric charge. One may conjecture that, in moving reference frame when photons are massive, an electric charge is seen to be equivalent to a stationary magnetic charge. The relation between the electric and magnetic charges is already obtained by Dirac [3].

Let us now write Maxwell’s equations, relating the electric and magnetic properties due to electric charge, and transform it under eq.(1). Applying eq.(1) in Maxwell’s equations yields the combined Maxwell’s equations due to electric and magnetic charges. Substituting eq.(1) in Gauss’s law reads

$$\alpha \left( \nabla \cdot \vec{E} - \rho_e \right) - i\lambda \left( \nabla \cdot \vec{B} - \rho_m \right) = 0,$$

while the divergence free magnetic field yields

$$\alpha \nabla \cdot \vec{B} + i\lambda \nabla \cdot \vec{E} = 0.$$

Applying eq.(1) in Faraday’s law yields

$$\alpha \left( \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) - i\lambda \left( \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \right) = 0,$$

and in Ampere’s law yields

$$\alpha \left( \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} - \mu_0 \vec{j}_e \right) + i\lambda \left( \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} + \vec{j}_m \right) = 0,$$
where we have expressed the charge and current densities in the complex form

$$\rho' = \alpha \rho_e - i \lambda \rho_m, \quad \vec{J}' = \alpha \vec{J}_e - i \lambda \vec{J}_m,$$

(12)

where $\vec{J}_e(\rho_e)$ and $\vec{J}_m(\rho_m)$ are the currents/charges densities due to the motion of electric and magnetic charges, respectively. Inasmuch as a moving electric charge produces electric current, a moving magnetic charge would analogously produce an electric current besides the magnetic current.

Now consider eqs.(8) and (11) and equate the real and imaginary parts in each side to zero. These yield Maxwell’s equation in free space

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \cdot \vec{E} = 0,$$

and

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

(13)

while eqs.(9) and (10) yield Maxwell’s equations in a medium, viz.,

$$\vec{\nabla} \cdot \vec{B} = \rho_m \quad \vec{\nabla} \cdot \vec{E} = \rho_e,$$

and

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}_e, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{J}_m.$$

(14)

The Maxwell’s equations in a medium are obtained from applying eq.(1) in Gauss and Ampere laws. It is interesting that the application of eq.(1) in the divergence free magnetic field and Faraday’s equations lead to Maxwell’s equations in free space.

3. DUALITY TRANSFORMATIONS

Under duality transformations, $\vec{E} \rightarrow c \vec{B}$ and $c \vec{B} \rightarrow -\vec{E}$ [8]. Maxwell’s equations in vacuum are invariant under these transformations. However, the invariance of Maxwell’s equations under these transformations in a medium requires the existence of a magnetic charge that was not present in the original Maxwell’s equations. Moreover, under duality transformations the electric and magnetic currents and charges densities are known to transform as $\rho_e(\rho_m) \rightarrow \rho_m(-\rho_e)$ and $\vec{J}_e(\vec{J}_m) \rightarrow \vec{J}_m(-\vec{J}_e)$ [8].

We have written in the foregoing sections Maxwell’s equations in a compact (complex) form that encompass these transformations. These equations describe the state of an electric and magnetic charges of the massive photon.

4. TWO COMPONENT FORMALISM

The electric and magnetic fields in eq.(1) can be expressed as (see the appendix)

$$\begin{pmatrix} \vec{E}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} \alpha & -i \lambda \\ i \lambda & \alpha \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}$$

(15)
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The two electromagnetic invariants are transformed as follows

\[ \vec{E}' \cdot \vec{B}' = a \vec{E} \cdot \vec{B} + ib (E^2 - B^2), \tag{16} \]

and

\[ E'^2 - B'^2 = -4ib \vec{E} \cdot \vec{B} + a (E^2 - B^2), \quad a = \lambda^2 + \alpha^2, \quad b = \lambda \alpha. \tag{17} \]

Equations (16) and (17) can be expressed as (see the appendix)

\[ \begin{pmatrix} \vec{E}' \cdot \vec{B}' \\ E'^2 - B'^2 \end{pmatrix} = \begin{pmatrix} a & ib \\ -4ib & a \end{pmatrix} \begin{pmatrix} \vec{E} \cdot \vec{B} \\ E^2 - B^2 \end{pmatrix}, \tag{18} \]

while eqs. (5) and (7)

\[ \begin{pmatrix} \vec{F}_e' \cdot \vec{F}_m' \\ \vec{F}_e' \times \vec{F}_m' \end{pmatrix} = \begin{pmatrix} \alpha & -i\lambda \\ i\lambda & \alpha \end{pmatrix} \begin{pmatrix} \vec{F}_e \cdot \vec{F}_m \\ \vec{F}_e \times \vec{F}_m \end{pmatrix}. \tag{19} \]

The energy conservation equation is given by [8]

\[ \frac{\partial}{\partial t} \frac{1}{2} (E^2 + B^2) + \vec{\nabla} \cdot (\vec{E} \times \vec{B}) = -\vec{E} \cdot \vec{J}. \tag{20} \]

It is shown by Dvoeglazov and Gonzalez that [1]

\[ \begin{pmatrix} \vec{E}' + i\vec{B}' \\ \vec{E}' - i\vec{B}' \end{pmatrix} = \begin{pmatrix} f(p) & 0 \\ 0 & f(-p) \end{pmatrix} \begin{pmatrix} \vec{E} + i\vec{B} \\ \vec{E} - i\vec{B} \end{pmatrix}, \tag{21} \]

where

\[ f(p) = 1 - \frac{\vec{S} \cdot \vec{p}}{m} + \frac{(\vec{S} \cdot \vec{p})^2}{E(E + m)}. \tag{22} \]

Equation (21) can be written as

\[ \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} f(p) & 0 \\ 0 & f(-p) \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \quad \Psi_\pm = \vec{E} \pm i\vec{B}. \tag{23} \]

A complex wavefunction of the photon was suggested by Silberstein [5]. Dvoeglazov and Gonzalez related \( \Psi_+ \) and \( \Psi_- \) to the two components of Winger wavefunction, representing left and right components [1]. We remark here that these wavefunctions, \( \Psi_+ \) and \( \Psi_- \), are related to the two states the massive photon has; one with momentum \( \vec{p} \) and the other with \(-\vec{p}\). Thus if, \( \Psi_+ \) defines the state of an electric charge, then \( \Psi_- \) define the state with a magnetic charge. In particular, under parity transformation, \( \vec{p} \rightarrow -\vec{p}, \vec{E} \rightarrow -\vec{E} \) and \( \vec{B} \rightarrow \vec{B} \), we find that \( \Psi_+ \rightarrow -\Psi_- \) and \( \Psi_- \rightarrow -\Psi_+ \), and that \( \Psi_\pm' \rightarrow -\Psi_\mp' \) and \( \Psi_\mp' \rightarrow -\Psi_\pm' \). Hence, eq. (23) is invariant under parity transformation. We see from eq. (23) that the two states are related by a rotation matrix whose inverse is obtained by replacing \( \vec{p} \) by \(-\vec{p}\). Thus the two wavefunctions are dual to each other. Each one is obtained
from the other by the parity transformation. Hence, the first wavefunction defines a left-handed helicity and the second one the right-handed helicity. The state of the photon at any instant is described by a linear combination of these two charges states. At any time, the photon can’t bear both electric and magnetic charges simultaneously. The uncertainty in measuring these two charges or physical quantities related to them is governed by the Heisenberg uncertainty relation, $\Delta q_m \Delta q_e \geq \frac{\hbar}{2}$. This is reflected in the Dirac quantization rule that $q_m q_e = \frac{n \hbar}{2}$, where $n = 1, 2, \cdots [3]$. Hence, the electromagnetic properties due to electric charge described by the real parts of eqs.(8) - (11), and those due to the magnetic charges are described by the imaginary parts. The states due to magnetic charge are orthogonal to those due to magnetic charge. If a state of an electric charge is described by $\vec{E}, \vec{B}$, and $\vec{p}$, the state for the magnetic charge at that instant will be described by the same state but with $-\vec{E}, \vec{B}$, and $-\vec{p}$. Recall that the two states can’t be measured at the same time. The energy conservation equation in this state reveals a new charge state moving in opposite direction. It is like the motion of an antiparticle with opposite charge. This may urge us to identify the magnetic charge as the antiparticle. This assertion is plausible since the antiparticle has the same mass as that of a particle.

5. CONCLUDING REMARKS

By assuming that the electric and magnetic fields for massive photon can be expressed in complex form, we have obtained a set of Maxwell’s equation in free space and another one in a Medium. This enlarges the duality symmetry between electric and magnetic phenomena. It suggests that photons are massive in a medium and massless outside (vacuum). We further associate two wavefunction with these fields. These two wavefunctions (states) are related by the parity transformation. If the first state defines an electric charge, the second one defines a magnetic charge (antiparticle).

Acknowledgments

I would like to thank the anonymous referee for his/her critical comments.

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Appendix

It is interesting to note that if the fields are transformed as

\[
\begin{pmatrix}
\vec{E}' \\
\vec{B}'
\end{pmatrix}
= U
\begin{pmatrix}
\vec{E} \\
\vec{B}
\end{pmatrix}
\]

for some transforming matrix \( U \), then the energy densities are transformed as

\[
\begin{pmatrix}
\vec{E}' \cdot \vec{B}' \\
E'^2 - B'^2
\end{pmatrix}
= S
\begin{pmatrix}
\vec{E} \cdot \vec{B} \\
E^2 - B^2
\end{pmatrix},
\]

for some transforming matrix \( S \). Note that \(|S| = |U|^2\). Hence, if the fields are transformed (rotated) by \( U \) the energy densities are transformed (rotated) by \( U^2 \).