International Journal of Engineering Innovation & Research
Volume 3, Issue 2, ISSN: 2277 – 5668

Approximation Theorems for The Solution of Stochastic Functional Differential Equations with Discontinuous Initial Data

Tagelsir A Ahmed
Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Khartoum, Sudan.
E-mail: tagelsir6@gmail.com

Van Casteren, J. A.
Department of Mathematics and Computer Science
University of Antwerp (UA), Middelheimlaan 1
2020 Antwerp, Belgium
.

***Abstract*—Here "Stochastic Functional Differential Equations(S.F.D.E's)" means "Delay Stochastic Differential Equations". In this work we have developed an Euler approximation scheme for the solution process of Stochastic Functional Differential Equation with possibly discontinuous initial data, and we have shown that this Euler scheme (under appropriate conditions) converges to the solution process as the mesh of the partition goes to zero.**

The approximation theorem which we have established gives us a method for approximating the solution of S.F.D.E's with possibly discontinuous initial data. Note that here we are considering S.F.D.E which includes both drift and diffusion coefficients. The present work on approximation is an extension of the work on approximation in [1] to include S.F.D.E's with both drift and diffusion coefficients. The work on approximation in [1] was suggested by Prof. Salah-E.A.Mohammed and it was done by Tagelsir A. Ahmed under the supervision of Prof. Salah-E.A.Mohammed.

Keywords*—*Approximation, delay, differential, equation, functional, ordinary, stochastic.

I. Introduction, Notations and Definitions

A. Introduction

Stochastic Functional Differential Equations serve as models of noisy physical processes whose time evolution depends on their past history. In physics, laser dynamics with delayed feedback is often investigated as well as the dynamics of noisy bistable systems with delay. In biophysics, stochastic equations are used to model delayed visual feedback systems or human postural way. Since the model equations are generally non-linear and do not allow for explicit solutions, there is a clear need for numerical approximation methods for solutions of delay stochastic equations.Early investigations in this direction were investigated in [1] and [7]. Recently this topic has gained more attention. Specific approximation methods studied include the Euler-Maruyama scheme and the θ- method with order of strong convergence 0.5 and the Milstein method with order of strong convergence 1. See the website of prof. Salah-E.A.Mohammed namely "sfde.math.siu.edu". Many models of physical phenomena happen to be stochastic equations. For many of these stochastic differential equations, we can not find the solution explicitly, but we can use suitable approximation method to get an approximate solution for our ordinary SDE, or our Delay SDE.

B. Notations and Definitions

The following notations and definitions will be used throughout this work.

$(\Omega, \mathcal{F}, P)$ is a probability space. a is a positive real number. $\{\mathcal{F}_t\}_{t\in[0,a]}$ is an increasing family of of sub-σ algebras of $\mathcal{F}$, each of which contains all null subsets of Ω. $\mathbf{N}$ is the set of natural numbers. $W : [0,a] \times \Omega \to \mathbf{R}$ is a normalized Brownian motion. If X is a topological space, denote by $\mathcal{B}(X)$ the Borel field of X. λ refers to the Lebesgue measure. Denote the Euclidean norm on $\mathbf{R}^n \quad (n \in \mathbf{N})$ by $|\cdot|$.

Let G be a Banach space and let $\mathcal{A}$ be a sub-σ algebra of $\mathcal{F}$ containing all subsets of measure zero in $\mathcal{F}$, then $\mathcal{L}^2(\Omega, \mathcal{A}, P; G)$ denotes the space of all functions $f : \Omega \to G$ which are $\mathcal{A} - \mathcal{B}(G)$ measurable and are such that $\int_\Omega \|f\|_G^2 \, dP < \infty$.

$L^2(\Omega, \mathcal{A}, P; G)$ denotes the Banach space (with $\|f\|_{L^2}^2 = \int_\Omega \|f(\omega)\|^2 \, dP$) of all equivalence classes of functions $f : \Omega \to G$ which are $\mathcal{A} - \mathcal{B}(G)$ measurable and are such that $\int_\Omega \|f\|_G^2 \, dP < \infty$. $L(\mathbf{R}^m, \mathbf{R}^n)$ $(m, n \in \mathbf{N})$ denotes the space of all linear maps. J refers to the interval $[-1, 0)$. $\mathcal{H}(J)$ or $\mathcal{B}(J)$ refer to the Borel field on J.

If $x : [-1, a] \times \Omega \to \mathbf{R}^n$ is a process then for each $t \in [0, a]$; $\omega \in \Omega$ define the following maps:

(i) $x_t : \Omega \to \mathcal{L}^2(J, \mathbf{R}^n)$ by $x_t(\omega)(s) = x(t + s, \omega)$ $\forall s \in J$ and almost all ω.
(ii) $x^* : [0, a] \times \Omega \to L^2(J, \mathbf{R}^n)$ by $x^*(t, \omega)(s) = x(t + s, \omega)$, $\forall s \in J$.
(iii) $\tilde{\phi}_x$, ϕ_x : $[0, a] \times \Omega \to \mathbf{R}^n$ by $\tilde{\phi}_x(t, \omega) = f(t, \omega, x(t, \omega), x_t(\omega))$ and $\phi_x(t, \omega) = g(t, \omega, x(t, \omega), x_t(\omega))$ where f and g are maps from $[0, a] \times \Omega \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n)$ to $\mathbf{R}^n$.

Define $\|(x(t), x_t)\|^2 = \|x(t)\|^2 + \|x_t\|^2$ for each $0 \le t \le a$.
Let $V \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P; \mathbf{R}^n)$, $\theta \in \mathcal{L}^2(J \times \Omega, \mathcal{H}(J) \otimes \mathcal{F}_0, \lambda \otimes \mathcal{P}; \mathbf{R}^n)$ and $f, g : [0, a] \times \Omega \times \mathbf{R}^n \times \mathcal{L}^2(J, \mathbf{R}^n) \to \mathbf{R}^n$. Then a process $x : [-1, a] \times \Omega \to \mathbf{R}^n$ is called a solution of the S.F.D.E.

$$x(t) = \begin{cases} V + \int_0^t f(u, x(u), x_u)du \\ \quad + \int_0^t g(u, x(u), x_u)dW(u) & 0 \le t \le a \\ \theta(t) & t \in J \end{cases} \tag{1}$$

if

(i) x is $\mathcal{B}([0, a]) \otimes \mathcal{F} - \mathcal{B}(\mathbf{R}^n)$ measurable.
(ii) For each $t \in [0, a]$, $x(t, \cdot)$ is $\mathcal{F}_t - \mathcal{B}(\mathbf{R}^n)$ measurable and for each $t \in J$, $x(t, \cdot)$ is $\mathcal{F}_0 - \mathcal{B}(\mathbf{R}^n)$ measurable.
(iii) $x \in \mathcal{L}^2([-1, a] \times \Omega, \mathcal{H} \times \mathcal{F}, \lambda \times P; \mathbf{R}^n)$,
(iv) x satisfies the S.F.D.E. (1).

It can be proved that the following conditions (which we shall refer to by conditions E) are sufficient for the existence of a unique solution of (1):

(i) $V \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P; \mathbf{R}^n)$.
(ii) $\theta \in \mathcal{L}^2(J \times \Omega, \mathcal{H} \otimes \mathcal{F}_0, \lambda \otimes P, \mathbf{R}^n)$.
(iii) $f, g : [0, a] \times \Omega \times \mathbf{R}^n \times \mathcal{L}^2(J, \mathbf{R}^n) \to \mathbf{R}^n$ are such that
  a) f and g are $\mathcal{B}([0, a]) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(L^2(J, \mathbf{R}^n)) - \mathcal{B}(\mathbf{R}^n)$ measurable.
  b) For each $t \in [0, a]$, $f(t, \cdot, \cdot, \cdot)$ and $g(t, \cdot, \cdot, \cdot)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathcal{L}^2(J, \mathbf{R}^n)) - \mathcal{B}(\mathbf{R}^n)$ measurable.
  c) There exists a constant K and $\zeta \in \mathcal{L}^2(\Omega, \mathcal{F}, P; \mathbf{R}^n)$ such that $|f(t, \omega, s, h)| + |g(t, \omega, s, h)| \le K(|s| + \|h\| + |\zeta(\omega)|)$ for $a.a\omega$ and for all $t \in [0, a]$; $s \in \mathbf{R}^n$ and $h \in \mathcal{L}^2(J, \mathbf{R}^n)$.
  d) There exists a constant K' such that for $a.a\omega$ $|f(t, \omega, s, h_1) - f(t, \omega, u, h_2)| + |g(t, \omega, s, h_1) - g(t, \omega, u, h_2)| \le K'(|s - u| + \|h_1 - h_2\|)$ $\forall t \in [0, a]$; $s, u \in \mathbf{R}^n$; $h_1, h_2 \in \mathcal{L}^2(J, \mathbf{R}^n)$.

A Cauchy partition π of an interval $[a, b]$ is an ordered (2m+1) tuple
$\pi = (t_1, t_2, \cdots, t_{m+1}; t_1, t_2, \cdots, t_m)$ such that $a = t_1 < t_2 < \cdots < t_{m+1} = b$ where $[t_i, t_{i+1}]$; $i = 1, 2, \cdots, m$ are the intervals of the partition and for each $i = 1, 2, \cdots, m$; t_i is the evaluation point of the interval $[t_i, t_{i+1}]$.
In the sequel we suppress the dependence on ω of the variables $V(\cdot), \theta(t, \cdot), x(t, \cdot)$ and $x_t(\cdot)$.

II. Approximation Lemma's and Theorems

The following lemma will be needed to establish our Approximation Theorem. We have established this lemma by suitable modifications of the corresponding lemma in McShane([6], chapter 5, Lemma (3.4)).

Lemma 1 (Approximation Lemma)**.** *Suppose that:*

(i) *Conditions E are satisfied with the same notations.*
(ii) *x is the solution of (1).*
(iii) *For each process $X : [0, a] \times \Omega \to \mathbf{R}^n$ with continuous sample paths and each process*

$X^* : [0,a] \times \Omega \to L^2(J, \mathbf{R}^n)$ with continuous sample paths, the processes $(t,\omega) \mapsto f(t, X(t,\omega), X^*(,\omega))$ and $(t,\omega) \mapsto g(t, X(t,\omega), X^*(,\omega))$ are continuous in probability almost surely in the interval $[0,a]$.

(iv) *To each Cauchy partition $\pi = (t_1, t_2, \ldots, t_{m+1}; t_1, t_2, \ldots, t_m)$ of $[0,a]$, there corresponds a process $x^\pi : [-1,a] \times \Omega \to \mathbf{R}^n$ with the following properties:*

a) *For almost all $\omega \in \Omega$, $x^\pi(\cdot,\omega)$ is constant on each of the intervals $[t_1,t_2), [t_2,t_3), \ldots, [t_{m-1},t_m), [t_m, t_{m+1}]$ of π.*

b) *$x^\pi(s,\omega) = \theta(s,\omega)$ for all $s \in J$ and almost all ω; $x^\pi(0,\omega) = V(\omega)$ for almost all ω.*

c) *For each $1 \le i \le m$, $x^\pi(t_i,\cdot)$ is $\mathcal{F}_{t_i} - \mathcal{B}(\mathbf{R}^n)$ measurable and $x^\pi_{t_i}$ is $\mathcal{F}_{t_i} - \mathcal{B}(L^2(J,\mathbf{R}^n))$ measurable.*

d) *Denote, with $\Delta_i = t_{i+1} - t_i$ and $\Delta_i W = W(t_{i+1}) - W(t_i)$,*

$$\begin{aligned}\epsilon^\pi(t_i) &= x^\pi(t_{i+1}) - x^\pi(t_i) \\ &\quad - f(t_i, x^\pi(t_i), x^\pi_{t_i})\Delta_i \\ &\quad - g(t_i, x^\pi(t_i), x^\pi_{t_i})\Delta_i W,\end{aligned}$$

where for each $\omega \in \Omega$ and $s \in J$, $x^\pi_t(\omega)(s) = x^\pi(t+s,\omega)$. And hence, there exists a positive valued function $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\phi(r) \to 0$ as $r \to 0$ and $\left\|\sum_{i=1}^k \epsilon^\pi(t_i)\right\| \le \phi(\text{mesh } \pi)(1 + \sup_{0\le t\le a} \|(x^\pi(t), x^\pi_t)\|)$, for $k = 1,2,\ldots,m$, where $\text{mesh}\pi = \max_{1\le i\le m}\{(t_{i+1} - t_i)\}$.

Then $(x^\pi(t), x^\pi_t)$ converges to $(x(t), x_t)$ in L^2-norm uniformly for $0 \le t \le a$ as mesh π tends to zero.

Proof: It can be proved that the S.F.D.E. (1) has a unique solution (because of conditions E) see [1]. Let $\pi = (t_1, t_2, \ldots, t_{m+1}; t_1, t_2, \ldots, t_m)$ be a Cauchy partition and for $1 \le k \le m$ define $\tilde{x}(t_k) = V + \sum_{i=1}^{k-1} f(t_i, \cdot, x(t_i), x_{t_i})\Delta_i + g(t_i, \cdot, x(t_i), x_{t_i})\Delta_i W$. (Note: $\Delta_i = t_{i+1} - t_i$, and $\Delta_i W = W(t_{i+1}) - W(t_i)$.) Let $M - 1$ be an upper bound for $\|(x(t), x_t)\|$ $0 \le t \le a$. Let $N(t) = \sup_{0\le s\le t} \|(x^\pi(s), x^\pi_s) - (x(s), x_s)\|$. It is clear that given $\epsilon > 0$ $\exists \delta_1 > 0$ such that if mesh $\pi < \delta_1$, then

$$\|\tilde{x}(t_k) - x(t_k)\| < \epsilon \tag{2}$$

(See [2], chapter 3, Theorem (2.3) and [1]). Also it can be shown that if $0 \le s \le t \le a$ then $\exists \delta_2 > 0$ such that

$$\text{if } |t-s| < \delta_2 \text{ then } \|x(t) - x(s)\| < \epsilon. \tag{3}$$

It is clear that $\exists \delta_3 > 0$ such that if $r < \delta_3$, then

$$\phi(r) < \min\left\{\frac{\epsilon}{M}, \frac{1}{2}\right\}. \tag{4}$$

Now suppose that mesh $\pi < \min\{\delta_1, \delta_2, \delta_3\}$ and fix $t \in [0,a]$.
Then $\exists k \in \{1,2,\ldots,m\}$ such that $x^\pi(t) = x^\pi(t_k)$ and hence it can be checked that

$$\begin{aligned}&x^\pi(t) - x(t) = \tilde{x}(t_k) - x(t_k) + x(t_k) - x(t) \\ &+ \textstyle\sum_{i=1}^{k-1} x^\pi(t_{i+1}) - x^\pi(t_i) - f(t_i, x^\pi(t_i), x^\pi_{t_i})\Delta_i \\ &- \textstyle\sum_{i=1}^{k-1} g(t_i, x^\pi(t_i), x^\pi_{t_i})\Delta_i W \\ &+ \textstyle\sum_{i=1}^{k-1} \left[f(t_i, x^\pi(t_i), x^\pi_{t_i}) - f(t_i, x(t_i), x_{t_i})\right]\Delta_i \\ &+ \textstyle\sum_{i=1}^{k-1} \left[g(t_i, x^\pi(t_i), x^\pi_{t_i}) - g(t_i, x(t_i), x_{t_i})\right]\Delta_i W\end{aligned} \tag{5}$$

Now by assumption (iv)(c) of this Lemma it is easy to see that $f(t_i, x(t_i), x_{t_i}), f(t_i, x^\pi(t_i), x^\pi_{t_i}), g(t_i, x(t_i), x_{t_i})$ and $g(t_i, x^\pi(t_i), x^\pi_{t_i})$ are

$\mathcal{F}_{t_i} - \mathcal{B}(\mathbf{R}^n)$ measurable for $i = 1,2,\ldots,m$ (see [1], the existence theorem and lemma (2.4)) and hence by using results from ([1], Remark (2.2)) together with ([2], chapter 3, Lemma (2.2)) it is easy to see that

$$\begin{aligned}&\left\|\sum_{i=1}^{k-1}\left[f(t_i, x(t_i), x_{t_i}) - f(t_i, x^\pi(t_i), x^\pi_{t_i})\right]\Delta_i\right\| \\ &+ \left\|\sum_{i=1}^{k-1}\left[g(t_i, x(t_i), x_{t_i}) - g(t_i, x^\pi(t_i), x^\pi_{t_i})\right]\Delta_i W\right\| \\ &\le K\left(\int_0^{t_k} N(s)^2 ds\right)^{\frac{1}{2}}\end{aligned} \tag{6}$$

where K is a constant. Also

$$1 + \sup_{0\le s\le t_k} \|(x^\pi(s), x^\pi_s)\| \le N(t) + M \quad (\text{as } t > t_k). \tag{7}$$

Thus using assumption (iv)(d) of this lemma and inequalities (4) and (7) we find that

$$\left\|\sum_{i=1}^{k-1}\epsilon^{\pi}(t_i)\right\| < \epsilon + \frac{N(t)}{2}. \tag{8}$$

Also it is easy to see that $|t - t_k| < \delta_2$ and hence

$$\|x(t_k) - x(t)\| < \epsilon. \tag{9}$$

Now by inserting estimates (2), (6), (8) and (9) in (5) we find that

$$\|x^{\pi}(t) - x(t)\|^2 \le 27\epsilon^2 + \frac{3}{4}N(t)^2 + 3K^2\int_0^{t_k} N(s)^2 ds. \tag{10}$$

Also it can easily be seen that

$$\|x_t^{\pi} - x_t\|^2 \le \int_0^t N(s)^2 ds \tag{11}$$

and hence

$$N(t)^2 \le 108\epsilon^2 + 2B\int_0^t N(s)^2 ds. \tag{12}$$

where $B = 6K^2 + 2$

It is easy to see that the map $t \mapsto N(t)$ is continuous from the right on $[0, a]$, and hence we find by applying Grönwall's Lemma to (12) that $N(t)^2 \le 108\epsilon^2 e^{2Bt}$ $(0 \le t \le a)$. Thus $N(a) \to 0$ as mesh $\pi \to 0$ and hence $(x^{\pi}(t), x_t^{\pi})$ converges to $(x(t), x_t)$ in L^2-norm uniformly for $t \in [0, a]$. ■

Lemma 2. *Suppose that*

(i) *Conditions E are satisfied with the same notations.*

(ii) *For a given Cauchy partition* $\pi = (t_1, t_2, \ldots, t_{m+1}; t_1, t_2, \ldots, t_m)$ *of* $[0, a]$, *define a process* $X^{\pi} : [0, a] \times \Omega \to \mathbf{R}^n$ *as follows:*

   a) *For* $s \in J$ $X^{\pi}(s) = \theta(s)$ *and* $X^{\pi}(0) = V$ *a.s..*

   b) *For each* t *such that* $t_i < t \le t_{i+1}$; $1 \le i \le m$, $X^{\pi}(t) = X^{\pi}(t_i) + f(t_i, X^{\pi}(t_i), X_{t_i}^{\pi})(t - t_i) + g(t_i, X^{\pi}(t_i), X_{t_i}^{\pi})(W(t) - W(t_i))$.

Then we have

(i) *For each* $t \in [0, a]$, $X^{\pi}(t)$ *is* $\mathcal{F}_t - \mathcal{B}(\mathbf{R}^n)$ *measurable and* X_t^{π} *is* $\mathcal{F}_t - \mathcal{B}(L^2(J, \mathbf{R}^n))$ *measurable.*

(ii) *For each* $t \in [0, a]$, $X^{\pi}(t) \in \mathcal{L}^2(\Omega, \mathbf{R}^n)$ *and* $X_t^{\pi} \in \mathcal{L}^2(\Omega, \mathcal{L}^2(J, \mathbf{R}^n))$.

(iii) $X^{\pi}|[t_i, t_{i+1}]$ *is a martingale* $\forall i$ *such that* $1 \le i \le m$.

(iv) *Moreover, if we define* $x^{\pi} : [-1, a] \times \Omega \to \mathbf{R}^n$ *by* $x^{\pi}(s) = \theta(s)$ *a.s. if* $s \in J$; $x^{\pi}(0) = V$ *a.s., and* $x^{\pi}(t) = X^{\pi}(\eta(t, \pi))$ *if* $0 \le t \le a$ *(where* $\eta(t, \pi) = \max\{\{\{t_1, t_2, \ldots, t_m\} \bigcap [0, t]\}\}$*) then for each* $t \in [0, a]$, $x^{\pi}(t)$ *is* $\mathcal{F}_t - \mathcal{B}(\mathbf{R}^n)$ *measurable and* x_t^{π} *is* $\mathcal{F}_t - \mathcal{B}(L^2(J, \mathbf{R}^n))$ *measurable.*

(v) *For each* k *such that* $1 \le k \le m$, $\sup_{0 \le t \le t_{k+1}} \|(X^{\pi}(t), X_t^{\pi})\|^2 \le 4(a + 1)\sup_{0 \le t \le t_{k+1}} \|(x^{\pi}(t), x_t^{\pi})\|^2 + \|\theta\|^2$.

Proof: To prove (i) it is sufficient to prove by induction (on $i = 1, 2, \ldots, m$) that if $0 \le t \le t_i$ then $X^{\pi}(t)$ is $\mathcal{F}_t - \mathcal{B}(\mathbf{R}^n)$ measurable and X_t^{π} is $\mathcal{F}_t - \mathcal{B}(L^2(J, \mathbf{R}^n))$ measurable. $X^{\pi}(t_1) = X^{\pi}(0)$ is $\mathcal{F}_0 - \mathcal{B}(\mathbf{R}^n)$ measurable. Now by using the fact that $\theta(s, \cdot)$ is $\mathcal{F}_0 - \mathcal{B}(\mathbf{R}^n)$ measurable it is easy to see that $X_0^{\pi} = \theta : \Omega \to \mathcal{L}^2(J, \mathbf{R}^n)$ is $\mathcal{F}_0 - \mathcal{B}(L^2(J, \mathbf{R}^n))$ measurable (see [1], Lemma (2.4)). Using the definition of X^{π} it is easy to see that the considered statement is true for $i = 1$.

Now suppose that for some k such that $1 \le k \le m$ and for all t such that $0 \le t \le t_k$, $X^{\pi}(t)$ is $\mathcal{F}_t - \mathcal{B}(\mathbf{R}^n)$ measurable, then by using condition E(iii)(b) it is easy to see that $X^{\pi}(t)$ is $\mathcal{F}_t$-measurable and X_t^{π} is $\mathcal{F}_t$-measurable $\forall 0 \le t \le t_{k+1}$. Thus (i) is proved.

To prove (ii) it is sufficient to prove by induction (on $i = 1, 2, \ldots, m$) that if $0 \le t \le t_i$ then $X^{\pi}(t) \in \mathcal{L}^2(\Omega, \mathbf{R}^n)$ and $X_t^{\pi} \in \mathcal{L}^2(\Omega, \mathcal{L}^2(J, \mathbf{R}^n))$. This statement can easily be proved using conditions E(i) and E(ii).

To prove part (iii) it is sufficient to prove that for $i = 1, 2, \ldots, m$; X^{π} satisfies

(a) $\mathbf{E}|X^{\pi}(t)| < \infty$ $\forall t_i \le t \le t_{i+1}$ $(i = 1, 2, \ldots, m)$.

(b) $\mathbf{E}[X^{\pi}(t)|\mathcal{F}_s] = X^{\pi}(s)$ $\forall t_i \le s \le t \le t_{i+1}$.

(c) $X^{\pi}|[t_i, t_{i+1}]$ is continuous from the right on $[t_i, t_{i+1}]$.

The assertions in (a) and (c) can easily be proved.

Assertion (b) can be proved by using induction on $i = 1, 2, \ldots, m$ to show that if $t_i \leq s \leq t \leq t_{i+1}$, then $\mathbf{E}[X^\pi(t)|\mathcal{F}_s] = X^\pi(s)$. Thus part (iii) is proved. To prove (iv) observe that by using the definition of x^π and part (i) of this lemma we find that $x^\pi(s)$ is $\mathcal{F}_0$-measurable for all $s \in J$ and $x^\pi(t)$ is $\mathcal{F}_t$-measurable $\forall 0 \leq t \leq a$. To prove that x_t^π is $\mathcal{F}_t - \mathcal{B}(L^2(J, \mathbf{R}^n))$ measurable $\forall 0 \leq t \leq a$. Fix $t \in [0, a]$, then if $t \geq 1$ it can easily be seen that x_t^π is $\mathcal{F}_t - \mathcal{B}(L^2(J, \mathbf{R}^n))$ measurable by using part (i) of this lemma. If $t < 1$ we need only to show that for any $h \in \mathcal{L}^2(J, \mathbf{R}^n)$ the maps $Y_1, Y_2 : \Omega \to \mathbf{R}^n$ given by $Y_1(\omega) = \int_{-1}^{-t} h(s)\theta(t+s, \omega)ds$, $Y_2(\omega) = \int_{-t}^{0} h(s)X^\pi(\eta(t+s, \pi))ds \quad (s \in J)$ are $\mathcal{F}_t - \mathcal{B}(\mathbf{R}^n)$ measurable. This can easily be shown by using ([1], Lemma (2.4)).

For the proof of part (v) observe that using part (iv) of this lemma we can see that

$$\sup_{0 \leq t \leq t_{k+1}} \mathbf{E}\left|X^\pi(t)\right|^2 \leq 4 \sup_{0 \leq t \leq t_{k+1}} \mathbf{E}\left|x^\pi(t)\right|^2. \quad (13)$$

Also it is clear that for $0 \leq t \leq t_{k+1}$ we have

$$\mathbf{E}\left\|X_t^\pi\right\|^2 \leq \left\|\theta\right\|^2 + a \sup_{0 \leq t \leq t_{k+1}} \mathbf{E}\left|X^\pi(t)\right|^2$$

and hence by (13) we find that

$$\sup_{0 \leq t \leq t+k+1} \mathbf{E}\left\|X_t^\pi\right\|^2 \leq \left\|\theta\right\|^2 + 4a \sup_{0 \leq t \leq t_{k+1}} \mathbf{E}\left|x^\pi(t)\right|^2. \quad (14)$$

Thus by using (13) and (14) the estimate in part (v) can easily be seen. ■

The following theorem is established by suitable modifications of the work of McShane ([6], chapter 5, Theorem (4.3)).

Theorem 3 (Approximation Theorem). *If assumptions (i), (ii) and (iii) of Lemma 1 and assumption (ii) of Lemma 2 are satisfied, then by using the same notation as in these assumptions we have that $(X^\pi(t), X_t^\pi)$ converges to $(x(t), x_t)$ in L^2-norm uniformly in $t \in [0, a]$ as mesh $\pi \to 0$.*

Proof: For each Cauchy partition π define a process $x^\pi : [-1, a] \times \Omega \to \mathbf{R}^n$ as in part (iv) of Lemma 2 then it is easy to see that x^π satisfies assumptions (iv)(a) and (iv)(b) of Lemma 1. Also by using part (iv) of Lemma 2 it is clear that x^π satisfies assumption (iv)(c) of Lemma 1. To show that x^π satisfies assumption (iv)(d) of Lemma 1, denote

$$\begin{aligned} \epsilon^\pi(t_i) &= x^\pi(t_{i+1}) - x^\pi(t_i) - f(t_i, x^\pi(t_i), x_{t_i}^\pi)\Delta_i \\ &- g(t_i, x^\pi(t_i), x_{t_i}^\pi)(W(t_{i+1}) - W(t_i)) \quad (15) \end{aligned}$$

$i = 1, 2, \ldots, m$, then we infer that

$$\begin{aligned} \epsilon^\pi(t_i) &= \left[f(t_i, X^\pi(t_i), X_{t_i}^\pi) - f(t_i, X^\pi(t_i), x_{t_i}^\pi)\right]\Delta_i \\ &+ \left[g(t_i, X^\pi(t_i), X_{t_i}^\pi) - g(t_i, X^\pi(t_i), x_{t_i}^\pi)\right]\Delta_i W. \end{aligned}$$

where $\Delta_i W = W(t_{i+1}) - W(t_i)$.

Now fix $t \in [0, a]$ then $\exists i$ such that $1 \leq i \leq m$ and $\eta(t, \pi) = t_i$. (Note: η is as defined in Lemma 2.) Then by using the definition of X^π and the linear growth condition on f and g and part (iv) and (v) of Lemma 2 it can be shown that

$$\begin{aligned} &\mathbf{E} \quad \left\|X_t^\pi - x_t^\pi\right\|^2 \\ &\leq \textstyle\sum_{j=1}^{i} KK'^2(1 + \left\|X^\pi(t_j)\right\|^2 + \left\|X_{t_j}^\pi\right\|^2)I \\ &\leq 4KK'^2(1 + \sup_{0 \leq t \leq t_{i+1}} \left\|(X^\pi(t), X_t^\pi)\right\|^2)pt_{i+1} \\ &\leq 4KK'^2 apL(1 + \sup_{0 \leq t \leq t_{i+1}} \left\|(x^\pi(t), x_t^\pi)\right\|^2) \end{aligned} \quad (16)$$

where $I = \int_{t_j}^{t_{j+1}}(r - t_j)dr$, $p =$ mesh π, K is constant and $L = \max\{1 + \left\|\theta\right\|^2, 4(a+1)\}$.

Using (15) and (16) we conclude that for $1 \leq k \leq m$,

$$\begin{aligned} &\left\|\textstyle\sum_{i=1}^{k} \epsilon^\pi(t_i)\right\| \\ &\leq \tfrac{1}{2}C\left\{\textstyle\sum_{i=1}^{k} \left\|f(t_i, X^\pi(t_i), X_{t_i}^\pi)\right.\right. \\ &\qquad \left.\left. - f(t_i, X^\pi(t_i), x_{t_i}^\pi)\right\|^2 (t_{i+1} - t_i)\right\}^{\frac{1}{2}} \\ &\quad + \tfrac{1}{2}C\left\{\textstyle\sum_{i=1}^{k} \left\|g(t_i, X^\pi(t_i), X_{t_i}^\pi)\right.\right. \\ &\qquad \left.\left. - g(t_i, X^\pi(t_i), x_{t_i}^\pi)\right\|^2 (t_{i+1} - t_i)\right\}^{\frac{1}{2}} \\ &\leq 2CDaK'\left(KL(\text{mesh } \pi)\right)^{\frac{1}{2}} \\ &\qquad \left(1 + \sup_{0 \leq t \leq t_{k+1}} \left\|(x^\pi(t), x_t^\pi)\right\|\right) \end{aligned}$$

where C is a constant (see [2], chapter 3, Lemma (2.2)).

Now define $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ by $\phi(r) = 2CDaK'(KLr)^{\frac{1}{2}}$ then it is easy to see that $\phi(r) \to 0$ as $r \to 0$ and hence x^π satisfies assumption (iv)(d) of Lemma 1. Thus x^π satisfies all the assumptions of Lemma 1 and hence $(x^\pi(t), x_t^\pi)$ converges in

L^2-norm to $(x(t), x_t)$ uniformly in $t \in [0, a]$. To complete the proof of the Approximation Theorem (3) it is sufficient to show that $\|(X^\pi(t), X_t^\pi) - (x^\pi(t), x_t^\pi)\| \leq B(\text{mesh } \pi)$ for some constant B. Now fix $t \in [0, a]$ then $\exists i$ such that $1 \leq i \leq m$ and $\eta(t, \pi) = t_i$, then using the definition of x^π and X^π we can see that

$$X^\pi(t) - x^\pi(t) = f(t_i, X^\pi(t_i), X_{t_i}^\pi)\Delta_i + g(t_i, X^\pi(t_i), X_{t_i}^\pi)(W(t) - W(t_i))$$

Then by taking norm on both sides of the above equation and using part (v) of Lemma 2 we find that for all $0 \leq t \leq a$

$$\|X^\pi(t) - x^\pi(t)\|^2 \leq 12KK'^2L(1 + \sup_{0\leq t\leq a} \|(x^\pi(t), x_t^\pi)\|^2)(\text{mesh } \pi). \tag{17}$$

As x^π converges to the solution as mesh $\pi \to 0$ then there exists a constant M such that $\sup_{0\leq t\leq a} \|x^\pi(t), x_t^\pi)\| \leq M$. Thus using (16) and (17) we can see that for each $t \in [0, a]$ $\|(X^\pi(t), X_t^\pi) - (x^\pi(t), x_t^\pi)\| \leq B(\text{mesh } \pi)$ where $B = 4KK'^2L(a + 3)(1 + M^2)$ is a constant. Thus the Approximation Theorem (3) is proved. ■

III. Further Approximation Theorem

The following remark and theorem show that the Approximation Lemma 1 and the Approximation Theorem 3 will also be valid when the field processes f and g are just locally Lipschitz instead of being globally Lipschitz on $\mathbf{R}^n \times L^2(J, \mathbf{R}^n)$.

Remark. The Approximation Lemma 1 will also be valid for the solution of the S.F.D.E.

$$x(t) = \begin{cases} V + \displaystyle\int_0^t f(u, x(u), x_u)du \\ \quad + \displaystyle\int_0^t g(u, x(u), x_u)dW(u) & 0 \leq t \leq a \\ \theta(t) & t \in J \end{cases} \tag{18}$$

if the following conditions are satisfied:

(i) $V \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P; \mathbf{R}^n)$.
(ii) $\theta \in \mathcal{L}^2(J \times \Omega, \mathcal{H}(J) \otimes \mathcal{F}_0, \lambda \otimes P; \mathbf{R}^n)$.
(iii) $f, g : [0, a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to \mathbf{R}^n$ satisfy:
 a) f and g are $\mathcal{B}([0.a]) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(L^2(J, \mathbf{R}^n) - \mathcal{B}(\mathbf{R}^n)$ measurable.
 b) There exists a constant K such that $\forall t \in [0, a]$; $v \in \mathbf{R}^n$; $h \in \mathcal{L}^2(J, \mathbf{R}^n)$ we have $|f(t, v, h)| + |g(t, v, h)| \leq K(|v| + \|h\| + 1)$.
 c) For each $N > 0$ there exists a constant K_N such that $\forall t \in [0, a]$, $\forall (v_i, h_i) \in \mathbf{R}^n \times L^2(J, \mathbf{R}^n)$ with $\|(v_i, h_i)\|^2 = |v_i|^2 + \|h_i\|^2 \leq N^2$ $(i = 1, 2)$, we have $|f(t, v_1, h_1) - f(t, v_2, h_2)| + |g(t, v_1, h_1) - g(t, v_2, h_2)| \leq K_N(|v_1 - v_2| + \|h_1 - h_2\|)$.
 d) If x^π is as in Lemma 1 then there exists a positive constant D_1 such that $\sup_{0\leq t\leq a} \|(x^\pi(t), x_t^\pi)\| \leq D_1$ for any partition π.

Proof of Remark: The proof of this remark is the same as that of the Approximation Lemma 1 except taht a slight modification is needed when dealing with the local Lipschitz condition. The following estimate will be needed to prove this remark:

$$\mathbf{E}\{\sup_{0\leq t\leq a} \|(x(t), x_t)\|^2 \leq S(\|V\|^2 + \|\theta\|^2 + 1)$$

where S is constant. ■

The following theorem is a strong version of the Approximation Theorem (3) and it can easily be proved using Theorem 3.

Theorem 4 (Strong Approximation Theorem)**.** *Suppose that*

(i) *Conditions (i), (ii) and (iii) of the above Remark and condition (iii) of the Approximation Lemma 1 are satisfied with the same notations and let x be the solution of (18).*
(ii) *For a given Cauchy partition $\pi = (t_1, t_2, \ldots, t_{m+1}; t_1, t_2 \ldots, t_m)$ of $[0, a]$ and for any $N > 0$ let $X^{N\pi} : [-1, a] \times \Omega \to \mathbf{R}^n$ be such that*
 a) *For each $N > 0$ and $s \in J$, $X^{N\pi}(s) = \theta(s)$ a.s., $X^{N\pi}(0) = V$ a.s..*
 b) *For $i = 1, 2, \ldots, m$ define $X^{N\pi}$ for*

$N > 0$; $t_i < t \leq t_{i+1}$ by

$$X^{N\pi}(t) = X^{N\pi}(t_i) + f_N(t_i, X^{N\pi}(t_i), X_{t_i}^{N\pi})\,(t - t_i) + g_N(t_i, X^{N\pi}(t_i), X_{t_i}^{N\pi})(W(t) - W(t_i))$$

where for each $N > 0$, $f_N, g_N : [0,a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to \mathbf{R}^n$ are defined by:

$$f_N(t,v,h) = \begin{cases} f(t,v,h) & \text{if } D \leq N. \\ f(t,v,h)(2 - \frac{D}{N}) & \text{if } N < D \leq 2N. \\ 0 & \text{if } D > 2N. \end{cases}$$

$$g_N(t,v,h) = \begin{cases} g(t,v,h) & \text{if } D \leq N. \\ g(t,v,h)(2 - \frac{D}{N}) & \text{if } N < D \leq 2N. \\ 0 & \text{if } D > 2N. \end{cases}$$

where $D = \|(v,h)\|$

Then $(X^{N\pi}(t), X_t^{N\pi})$ converges to $(x(t), x_t)$ in L^2-norm uniformly for $t \in [0,a]$ as mesh $\pi \to 0$ and $N \to \infty$.

IV. Remarks:

(a) All the results which we have established in this work can be extended by replacing the Brownian motion W by another process $Z : [0,a] \times \Omega \to \mathbf{R}$ which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant K the inequalities

$$|\mathbf{E}[Z(t) - Z(s)]|\mathcal{F}_s| \leq K(t-s) \quad \text{and}$$
$$\mathbf{E}\left(|Z(t) - Z(s)|^2\,|\mathcal{F}_s\right) \leq K(t-s)$$

for $0 \leq s \leq t \leq a$.

Observe that the above properties of Z which we have just mentioned are the only properties of W which we have used (in case of Brownian motion) to prove the results which we have obtained in this work.

(b) The Approximation Lemma's 1 and 2 and the Approximation Theorem 3 and the Strong Approximation Theorem 4, can be extended to processes $f', g' : [0,a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to L(\mathbf{R}^m, \mathbf{R}^n)$ $(m, n \in \mathbf{N})$ instead of the processes $f, g : [0,a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to \mathbf{R}^n$ $(n \in \mathbf{N})$, and instead of the Brownian motion W we use the process $Z : [0,a] \times \Omega \to \mathbf{R}^m$ which is a martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$, continuous on $[0,a]$, and has independent increments and satisfies for some constant K the inequalities

$$|\mathbf{E}[Z(t) - Z(s)]|\mathcal{F}_s| \leq K(t-s) \quad \text{and}$$
$$\mathbf{E}\left(|Z(t) - Z(s)|^2\,|\mathcal{F}_s\right) \leq K(t-s)$$

for $0 \leq s \leq t \leq a$.

(c) All the lemmas and theorems in this work hold for any delay interval $J' = [-r, 0)$ $(r \geq 0)$.

References

[1] Tagelsir A Ahmed, *Stochastic Functional Differential Equations with Discontinuous Initial Data, M.Sc. Thesis, University of Khartoum, Khartoum, Sudan,* (1983).

[2] Friedman, A., *Stochastic Differential Equations and Applications,* Academic press (1975).

[3] Halmos, P.R., *Measure Theory,* D. Van Nostrand Company (1950).

[4] Ikeda, N. and Watanabe, S, *Stochastic differential equations and Diffusion Processes,* Amsterdam: North-Holland 1981.

[5] Kloeden, P.E and Platen,E., *Numerical Solution of Stochastic Differential Equations,* Springer-Verlag Berlin Heidelberg (1992).

[6] McShane, E.J., *Stochastic calculus and Stochastic Models,* Academic Press (1974).

[7] Mohammed, S.E.A., *Stochastic Functional Differential Equations,* Research Notes in Mathematics; Pitman Books Ltd., London (1984).